Time : 2 hours Max Marks : 40

DIFFERENTIAL GEOMETRY II - MID-SEMESTRAL EXAM.

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) Decide whether the following statements are *True* or *False*. Answers without correct and complete justifications will not be awarded any points.

(a) Given a manifold M, $p \in M$ and $v \in T_p(M)$, there exists a compactly supported vector field X on M with $X_p = v$.

(b) Let A be a real $n \times n$ symmetric matrix and $c \in \mathbb{R}$. Then

$$M = \left\{ x \in \mathbb{R}^n \, : \, x^t A x = c \right\}$$

is a submanifold of $\mathbb{R}^n$.

(c) On a compact manifold every smooth real valued function has at least two critical points.

(d) The function $f : S^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y, z) = xy$$

has exactly two critical points. [4x4=16]

(2) (a) Define the Lie bracket of two vector fields. Show that the Lie bracket of two smooth vector fields is smooth.

(b) Let $X, Y \in \mathfrak{X}(M)$ be two smooth vector fields on a manifold M. If $X(f) = Y(f)$ for all $f \in C^\infty(M)$, show that $X = Y$.

(3) (a) Let X be the vector field on $\mathbb{R}^2$ defined by

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Find the maximal integral curve of X through $p = (a, b)$. Is X complete when thought of as a vector field on $\mathbb{R}^2 - 0$, on $\mathbb{R}^2 - (1, 0)$? [6+6]

(b) Let X be a smooth vector field on $\mathbb{R}^2$ satisfying

$$\left[\frac{\partial}{\partial x}, X\right] = X = \left[X, \frac{\partial}{\partial y}\right].$$

Determine X. [6+6]